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Single and Binary Black Holes and their Influence on Nuclear Structure

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Abstract

Massive central objects affect both the structure and evolution of galactic nuclei. Adiabatic growth of black holes generates power-law central density profiles with slopes in the range $1.5 \lesssim -d\log\rho/d\log r \lesssim 2.5$, in good agreement with the profiles observed in the nuclei of galaxies fainter than $M_V \approx -20$ mag. However, the shallow nuclear profiles of bright galaxies require a different explanation. Binary black holes are an inevitable result of galactic mergers, and the ejection of stars by a massive binary displaces a mass of order the binary's own mass, creating a core or shallow power-law cusp. This model is at least crudely consistent with core sizes in bright galaxies. Uncertainties remain about the effectiveness of stellar- and gas-dynamical processes at inducing coalescence of binary black holes, and uncoalesced binaries may be common in low-density nuclei. Numerical N-body experiments are not well suited to probing the long-term evolution of black hole binaries due to spurious relaxation.

1.1 Introduction

The effect of a supermassive black hole on its stellar surroundings depends, as so often in stellar dynamics, on how one imagines the system evolved to its present state. Collisional relaxation times are too long to have affected the stellar distribution in all but the densest nuclei (Faber et al. 1997); hence, the structure and kinematics of nuclei are fossil relics of the interactions between stars and black holes. In the simplest scenario, the black hole grows by accreting gas on a time scale long compared with the orbital periods of the surrounding stars (Peebles 1972; Young 1980). This "adiabatic growth" model makes fairly definite predictions about the distribution of stars near the black hole, predictions that are consistent with the steep density cusps observed in faint galaxies but that cannot explain the flatter profiles at the centers of bright galaxies. But galaxies merge, implying the formation of binary black holes (Begelman, Blandford, & Rees 1980) that are efficient at displacing matter as they spiral together. The dynamics of black hole binaries in galactic nuclei are complex; among the unanswered questions are the long-term efficiency of stellar dynamical processes at extracting energy from a binary, and whether decay of black hole binaries ever stalls at separations too great for the efficient emission of gravitational waves. But the binary black hole model is at least crudely consistent with the observed dependence of nuclear structure on galaxy luminosity. This article summarizes theoretical work on the single and binary black hole models and suggests avenues for future progress.

1.2 Preliminaries

A black hole of mass $M_\bullet$ embedded in a galactic nucleus will strongly affect the motion of stars within a distance $r = r_h$, the "radius of influence." A standard definition for r_h is

$$r_h \equiv \frac{GM_\bullet}{\sigma^2} \approx 10.8 \text{ pc} \left(\frac{M_\bullet}{10^8 M_\odot}\right)\left(\frac{\sigma}{200 \text{ km s}^{-1}}\right)^{-2} \tag{1.1}$$

with σ the 1D velocity dispersion of the stars at $r \gg r_h$. This definition had its origin in the isothermal sphere model for galactic nuclei; σ is independent of r in such a model and r_h is the radius at which the circular velocity around the black hole equals σ. We now know that nuclei are power laws in the stellar density, $\rho \sim r^{-\gamma}$, and that γ can lie anywhere between ~ 0 and ~ 2.5 (Lauer, this volume). The velocity dispersion in a power-law nucleus is only constant if $\gamma = 0$ or 2, hence a definition like equation (1.1) is problematic. One alternative would be to define r_h as the root of $\sigma^2(r) - GM_\bullet/r = 0$. A simpler definition, which will be adopted in this article, is the radius at which the enclosed mass in stars is twice the black hole mass:

$$M_*(r < r_h) = 2M_\bullet. \tag{1.2}$$

This definition is exactly equivalent to equation (1.1) when $\rho(r) = \sigma^2/2\pi G r^2$, the singular isothermal sphere.

1.3 The Adiabatic Growth Model

If a black hole grows at the center of a stellar system through the accretion of gas, the stellar density in the core will also grow as the black hole's gravity pulls in nearby stars (Peebles 1972; Young 1980). The change in the stellar density can be computed straightforwardly if it is assumed that the black hole grows on a time scale long compared with stellar orbital periods. This is reasonable, since even Eddington-limited accretion requires $\sim 10^8$ yr to double the black hole mass, and orbital periods throughout the region dominated by the black hole are $\lesssim 10^6$ yr. Under these assumptions, the adiabatic invariants $\mathbf{J}$ associated with the stellar orbits are conserved as the black hole grows, and the phase-space density f remains fixed when expressed in terms of the $\mathbf{J}$. Computing the final f becomes a simple matter of expressing the final orbital integrals in terms of their initial values under the constraint that the adiabatic invariants remain fixed (Young 1980).

In spherical potentials, the adiabatic invariants are the angular momentum L and the radial action $I = 2\int_{r_-}^{r_+} \sqrt{2[E-\Phi(r)] - L^2/r^2}$, where $\Phi(r)$ is the gravitational potential and $r_\pm$ are pericenter and apocenter radii. It may be shown (e.g., Lynden-Bell 1963) that orbital shapes remain nearly unchanged when L and I are conserved, implying that an initially isotropic velocity distribution $f(E)$ remains nearly isotropic after the black hole grows (though not exactly isotropic – see below). The final f corresponding to an initially isotropic f is then simply

$$\begin{aligned} f_f(E_f, L) &= f_i(E_i, L) \\ &\approx f_f(E_f) \end{aligned} \tag{1.3}$$

where E_f is related to E_i through the condition $I_f(E_f, L) = I_i(E_i, L)$.

While the form of the nuclear density profile before the black hole appeared is not known,

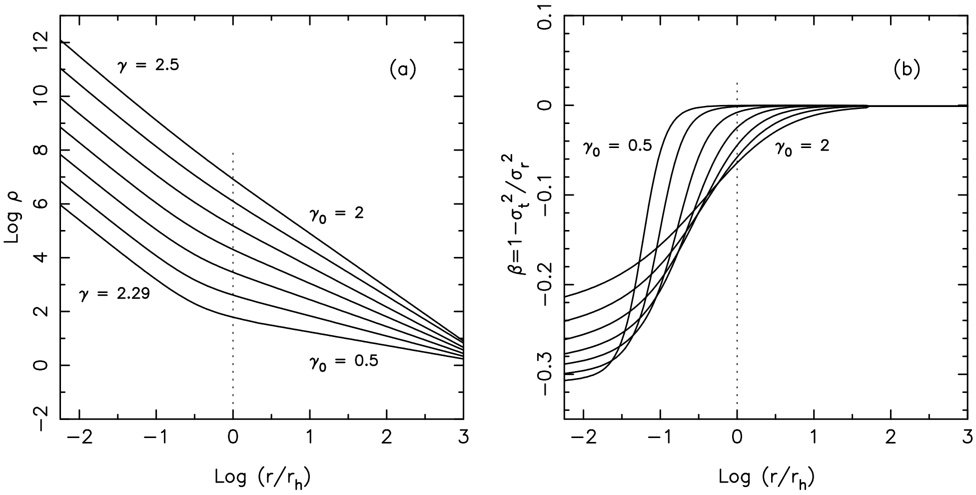

Fig. 1.1. Influence of the adiabatic growth of a black hole on its nuclear environment in a spherical, isotropic galaxy. (*a*) Density profiles after growth of the black hole. Initial profiles were power laws, $\rho_i \propto r^{-\gamma_0}$, with γ_0 increasing upwards in steps of 0.25. The radial scale is normalized to r_h as defined in the initial galaxy (Eq. 1.2). The slope of the final profile at $r < r_h$ is almost independent of the initial slope. (*b*) Velocity anisotropies after growth of the black hole. A slight bias toward circular motions appears at $r < r_h$.

N-body studies of structure formation suggest that power laws are generic (Power et al. 2003 and references therein) and observed nuclear density profiles are often well described as power laws even on scales $r \gg r_h$. Setting $\rho_i \propto r^{-\gamma_0}$, $\Phi_i \propto r^{2-\gamma_0}$ $(0 < \gamma_0 < 2)$, the initial distribution function becomes

$$f_i(E_i) \propto E_i^{-\beta}, \quad \beta = \frac{6-\gamma_0}{2(2-\gamma_0)} \quad (0 < \gamma_0 < 2). \tag{1.4}$$

To compute $f_f(E_f)$ we need a relation between E_f and E_i; we restrict attention to the region within the black hole's sphere of influence by setting $E_f = \upsilon^2/2 - GM_\bullet/r$. The radial action $I(E,L)$ in the power-law model cannot be computed analytically for every (E,L), but for certain orbits $E_f(E_i)$ has a simple form. For instance, circular orbits have $I = 0$, and conservation of angular momentum implies $r_iM_i(r_i) = r_fM_\bullet$ or $r_f \propto r_i^{4-\gamma_0}$. Thus $E_f \propto -r_f^{-1} \propto -r_i^{\gamma_0-4} \propto -E_i^{(\gamma_0-4)/(2-\gamma_0)}$, or

$$E_i \propto (-E_f)^{-(2-\gamma_0)/(4-\gamma_0)}. \tag{1.5}$$

The same relation turns out to be precisely correct for radial orbits as well and is nearly correct at intermediate eccentricities (Gondolo & Silk 1999). Thus we can write

$$f_f(E_f) = f_i(E_i) \propto E_i^{-\beta} \propto (-E_f)^{\delta}, \quad \delta = \frac{6-\gamma_0}{2(4-\gamma_0)} \;\; (0 < \gamma_0 < 2), \tag{1.6}$$

and the final density profile within the sphere of influence of the black hole is

$$\rho_f(r) \quad = \quad \int f_f(\upsilon) d^3\upsilon \propto \int_{\Phi(r)}^0 (-E)^{\delta} \sqrt{E-\Phi(r)}\, dE$$

$$\propto \quad r^{-\gamma}, \quad \gamma = 2 + \frac{1}{4-\gamma_0}. \tag{1.7}$$

For $0 < \gamma_0 < 2$, γ varies only between 2.25 and 2.5; the slope of the final density profile within r_h is almost independent of γ_0.

The form of $\rho_f(r)$ at $r \approx r_h$ must be computed numerically (e. g., Young 1980; Cipollina & Bertin 1994; Cipollina 1995; Quinlan, Hernquist, & Sigurdsson 1995; Gondolo & Silk 1999). Figure 1.1 shows $\rho_f(r)$ when $\rho_i(r) \propto r^{-\gamma_0}$. Defining r_{cusp} to be the radius at which the inner and outer power laws intersect, one finds

$$r_{cusp} = \alpha r_h, \quad 0.19 \lesssim \alpha \lesssim 0.22, \quad 0.5 \leq \gamma_0 \leq 1.5. \tag{1.8}$$

In early treatments of the adiabatic growth model (Peebles 1972; Young 1980), the black hole was assumed to grow inside of a constant-density isothermal core. The index of the power-law cusp that forms from this initial state is $\gamma = 1.5$, compared with the limiting value $\gamma = 2.25$ as $\gamma_0 \to 0$ in the power-law models. This difference can be traced to differences in the central density profile:

$$\rho_i(r) = \rho_0 \times \left(1 + C_1 r + C_2 r^2 + \ldots\right). \tag{1.9}$$

The isothermal model has $C_1 = 0$ (an "analytic core") implying a phase-space density that tends to a constant value at low energies. Other sorts of cores have $C_1 \neq 0$ and f diverges at low energies; for instance, the core produced by setting $\gamma = 0$ in $\rho(r) = r^{-\gamma}(1+r)^{-4}$ has $f(E) \to [E - \Phi(0)]^{-1}$. In fact models with finite central ρ's can be found that generate final cusp slopes anywhere in the range $1.5 \leq \gamma \leq 2.25$ (Quinlan et al. 1995). There is probably no way of ruling out an analytic core in the progenitor galaxy on the very small scales that are relevant to the later formation of a cusp, hence the adiabatic growth model is compatible with any final slope in the range $1.5 \lesssim \gamma \lesssim 2.5$. The upper limit could even be extended beyond 2.5 if $\gamma_0 > 2$.

How do these predictions compare with the data? Observed luminosity profiles are well described as power laws at the smallest resolvable radii, and in the case of faint ellipticals, $M_V \gtrsim -20$ mag, the observed range of slopes is $1.5 \lesssim \gamma \lesssim 2.5$ (Lauer, this volume). This is precisely the range in γ predicted by the adiabatic growth model. However, in bright galaxies, γ extends down to ~ 0. A natural interpretation is that the steep cusps in faint galaxies are a result of adiabatic black hole growth, while some additional mechanism, like mergers, has acted to modify the profiles in the brighter galaxies.

Some fine tuning is still required to reproduce the luminosity profiles of galaxies with $1.5 \lesssim \gamma \lesssim 2$. These intermediate slopes require a shallow, core-like initial profile, and if the initial core radius exceeded $\sim r_h$, the final profile will exhibit an upward inflection at $r \approx r_h$ (e.g., Fig. 2 of Young 1980; Fig. 1.1). Such inflections are rarely, if ever, seen; observed profiles have slopes that decrease smoothly inward. A way out is to require that the black hole mass exceed the initial core mass so that its growth obliterates the core; alternatively, all nuclei with $\gamma \lesssim 2$ may have been the products of mergers.

An ingenious attempt to reconcile the adiabatic growth model with both steep and shallow cusps was made by van der Marel (1999). He postulated the existence of isothermal cores in the progenitor galaxies with core masses scaling as $\sim L^{1.5}$, with L the total galaxy luminosity. Since $M_\bullet \sim L$, $M_\bullet/M_{core} \sim L^{-0.5}$ and the black holes in faint galaxies would grow to dominate their cores, producing a cusp profile that approximates the featureless power laws of faint galaxies. In bright galaxies, van der Marel argued that the upward inflection at

$r \lesssim r_h$ would be difficult to resolve; hence these galaxies would exhibit nearly unperturbed core profiles. Van der Marel's model is intriguing, although the assumed relation between core size and galaxy luminosity is ad hoc, and the assumption of large, pre-existing cores does not fit naturally into any current model of galaxy formation.

Black hole growth in axisymmetric nuclei with analytic cores was considered by Leeuwin & Athanassoula (2000). They found little dependence of the final cusp slope on the degree of flattening. Merritt & Quinlan (1998) grew black holes of various masses in a triaxial N-body model that was formed via gravitational collapse, and found $\rho \sim r^{-2}$ at $r \lesssim r_h$. Triaxial potentials support a wide range of different orbit families and the effects of black hole growth on such models have not been examined in detail.

In the spherical geometry, adiabatic growth of a black hole induces a mild anisotropy in the stellar motions at $r \lesssim r_h$ due to the slightly different ways that circular and eccentric orbits respond to the changing potential; the net effect is a decrease in the average orbital eccentricity (Young 1980; Goodman & Binney 1984; Quinlan et al. 1995; Fig. 1.1). If the progenitor galaxy is rotating, growth of the black hole tends to increase V_{rot} more rapidly than σ (Lee & Goodman 1989; Leeuwin & Athanassoula 2000), although again the effect is slight.

To summarize: the adiabatic growth model is limited in its ability to reproduce the full range of luminosity profiles observed in galactic nuclei. The model predicts power-law profiles at $r \lesssim r_h$ with logarithmic slopes $1.5 \lesssim \gamma \lesssim 2.5$. This nicely brackets the range of slopes observed in the nuclei of galaxies fainter than $M_V \approx -20$ mag. However, slopes less than $\gamma \approx 1.5$ are not naturally produced by the adiabatic growth model, and some fine tuning is required to avoid inflections in the profile at $r \approx r_h$.

1.4 The Binary Black Hole Model

The adiabatic growth model was proposed (Peebles 1972) before the importance of galaxy interactions and mergers was appreciated. We now know that supermassive black holes have been present in at least some galaxies since redshifts of $z \approx 6$ (e.g., Fan et al. 2001), and we believe that most galaxies have experienced at least one major merger since that time; indeed the era of peak quasar activity may coincide with the era of galaxy assembly via mergers (e.g., Cavaliere & Vittorini 2000). If a nucleus forms via the merger of two galaxies containing pre-existing black holes, the net effect on the nuclear density profile is roughly the opposite of what the adiabatic growth model predicts: the black holes *displace* matter as they spiral into the center (Fig. 1.2). This is a natural way to account for the shallow nuclear profiles in bright galaxies. The process may be understood as a sort of dynamical friction, with the "heavy particles" (the black holes) transferring their kinetic energy to the "light particles" (the stars). However, most of the energy transfer takes place after the two black holes have come within each other's sphere of influence, and in this regime, the interaction with the background is dominated by another mechanism, the *gravitational slingshot* (Saslaw, Valtonen, & Aarseth 1974). The massive binary ejects passing stars at high velocities, removing them from the nucleus and simultaneously increasing its binding energy.

We begin by reviewing the dynamics of massive binaries in fixed, homogeneous backgrounds, then consider the more difficult problem of a binary located at the center of an inhomogeneous and evolving galaxy.

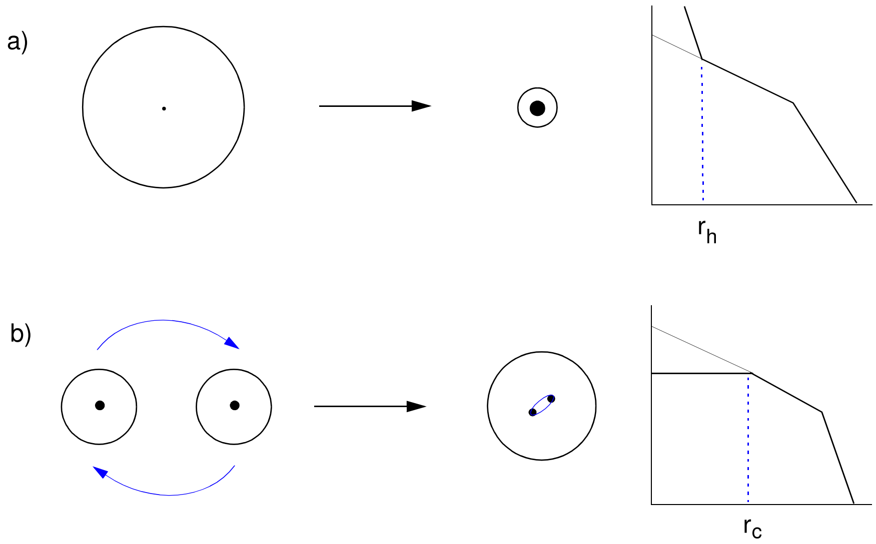

Fig. 1.2. Two schemes for growing black holes at the centers of galaxies. (*a*) The adiabatic growth model: the stellar density within r_h is increased as the black hole pulls in stars. (*b*) The binary black hole model: the inspiraling black holes displace matter within a distance r_c that is roughly the separation between the two black holes when they first form a bound pair.

1.4.1 Dynamics of Massive Binaries

Consider a binary system consisting of two black holes with masses M_1 and M_2, where $p \equiv M_2/M_1 \leq 1$ and $M_{12} \equiv M_1 + M_2$. Let a be the semi-major axis of the Keplerian orbit and e the orbital eccentricity. The binding energy of the binary is

$$|E| = \frac{GM_1M_2}{2a} = \frac{G\mu M_{12}}{2a}, \tag{1.10}$$

with $\mu = M_1M_2/M_{12}$ the reduced mass. The relative velocity of the two black holes, assuming a circular orbit, is

$$V_{bin} = \sqrt{\frac{GM_{12}}{a}} = 658 \text{ km s}^{-1} \left(\frac{M_{12}}{10^8 M_\odot}\right)^{1/2} \left(\frac{a}{1 \text{ pc}}\right)^{-1/2}. \tag{1.11}$$

A binary is called "hard" when its binding energy per unit mass, $|E|/M_{12} = G\mu/2a$, exceeds $\sim \sigma^2$. For concreteness, a binary will here be called hard if its separation falls below a_h, where

$$a_h \equiv \frac{G\mu}{4\sigma^2} \approx 0.27 \text{ pc} (1+p)^{-1} \left(\frac{M_2}{10^7 M_\odot}\right) \left(\frac{\sigma}{200 \text{ km s}^{-1}}\right)^{-2}. \tag{1.12}$$

Other definitions of a_h are possible (e. g. Hills 1983; Quinlan 1996).

Stars passing within a distance $\sim 3a$ of the center of mass of a hard binary undergo a complex interaction with the two black holes, followed almost always by ejection at velocity $\sim \sqrt{\mu/M_{12}} V_{bin}$ (Saslaw et al. 1974). Each ejected star carries away energy and angular momentum, causing the semi-major axis, eccentricity and orientation of the binary to change and the local density of stars to drop. If the stellar distribution is assumed fixed far from the binary and if the contribution to the potential from the stars is ignored, the rate at which these changes occur can be computed by carrying out scattering experiments of massless stars against a binary whose orbital elements remains fixed during each interaction.

The results of the scattering experiments can be summarized via a set of dimensionless coefficients $H, J, K, L, \ldots$ which define the mean rates of change of the parameters characterizing the binary and the stellar background (Hills & Fullerton 1980; Roos 1981; Hills 1983, 1992; Baranov 1984; Mikkola & Valtonen 1992; Quinlan 1996; Merritt 2001, 2002). These coefficients are functions of the binary mass ratio, eccentricity and hardness but are typically independent of a in the limit that the binary is very hard. The hardening rate of the binary is given by

$$\frac{d}{dt}\left(\frac{1}{a}\right) = H\frac{G\rho}{\sigma} \tag{1.13}$$

with ρ the density of stars at infinity. The mass ejection rate is

$$\frac{dM_{ej}}{d\ln(1/a)} = JM_{12} \tag{1.14}$$

with M_{ej} the mass in stars that escape the binary. The rate of change of the binary's orbital eccentricity is

$$\frac{de}{d\ln(1/a)} = K. \tag{1.15}$$

The diffusion coefficient describing changes in the binary's orientation is

$$\langle\Delta\vartheta^2\rangle = L\frac{m_*}{M_{12}}\frac{G\rho a}{\sigma}, \tag{1.16}$$

with m_* the stellar mass. Additional coefficients describe the rate of diffusion of the binary's center of mass, or "Brownian motion" (Merritt 2001).

The binary hardening coefficient H reaches a constant value of ~ 16 in the limit $a \ll a_h$, with a weak dependence on M_2/M_1 (Hills 1983; Mikkola & Valtonen 1992; Quinlan 1996). In a fixed background, equation (1.13) therefore implies that a hard binary hardens at a constant rate:

$$\frac{1}{a(t)} - \frac{1}{a_h} \approx H\frac{G\rho}{\sigma}(t - t_h), \quad t \geq t_h, \quad a(t_h) = a_h. \tag{1.17}$$

If the supply of stars remains steady, hardening continues at this rate until the components of the binary come close enough together that the emission of gravitational radiation is important. In this regime, gravity wave coalescence takes place in a time:

$$\begin{aligned} t_{gr} &= \frac{5}{256F(e)}\frac{c^5}{G^3\mu M_{12}^2}a^4, \\ F(e) &= (1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \end{aligned} \tag{1.18}$$

(Peters 1964). Coalescence in a time t_{gr} occurs when $a = a_{gr}$, where

$$\frac{a_h}{a_{gr}} \approx 75F^{-1/4}\frac{p^{3/4}}{(1+p)^{3/2}}\left(\frac{\sigma}{200\ \mathrm{km\ s^{-1}}}\right)^{-7/8}\left(\frac{t_{gr}}{10^9\mathrm{yr}}\right)^{-1/4}. \tag{1.19}$$

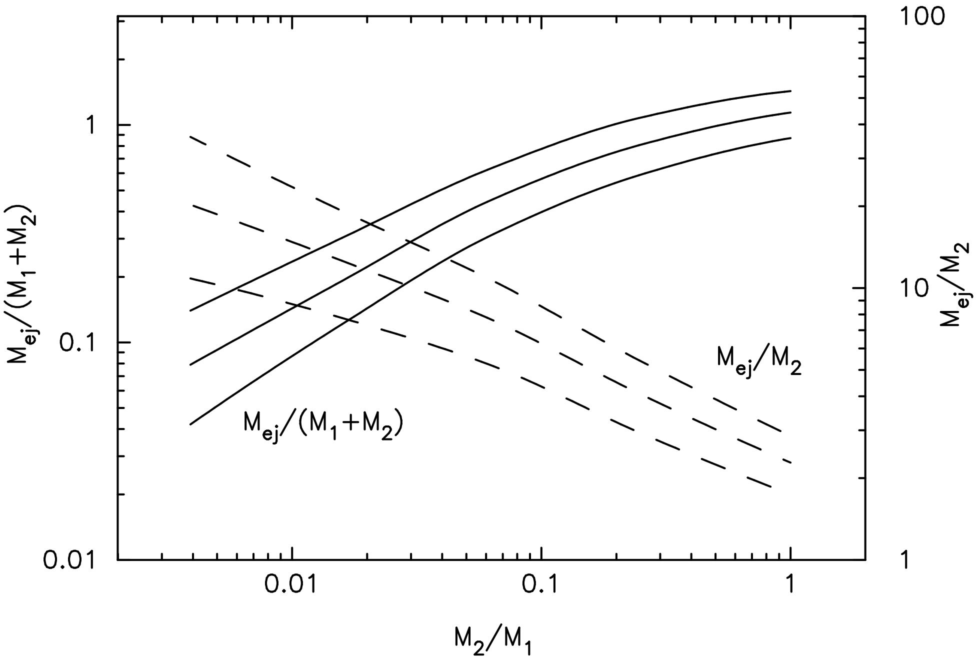

Fig. 1.3. Mass ejected by a decaying binary, in units of $M_{12} = M_1 + M_2$ (solid lines) or M_2 (dashed lines). Curves show mass that must be ejected in order for the binary to reach a separation where the emission of gravity waves causes coalescence on a time scale of 10^{10} yr (lower), 10^9 yr (middle) and 10^8 yr (upper).

The $M_\bullet - \sigma$ relation, $M_{12}/10^8 M_\odot \approx (\sigma/200 \text{ km s}^{-1})^{4.5}$, has been used to express M_{12} in terms of σ. For mass ratios p of order unity and $e \approx 0$, the binary must decay by a factor of $\sim 10^2$ in order for gravitational radiation to induce coalescence in a time shorter than 10^9 yr. Less decay is required if the binary is eccentric or if $M_2 \ll M_1$.

If the binary manages to shrink by such a large factor, the damage done to its stellar surroundings will be considerable. The mass ejected by the binary in decaying from a_h to a_{gr} is given by the integral of equation (1.14):

$$M_{ej} = M_{12} \int_{a_{gr}}^{a_h} \frac{J(a)}{a} da. \tag{1.20}$$

Figure 1.3 shows M_{ej} as a function of the mass ratio M_2/M_1 for $\sigma = 200$ km s^{-1} and various values of t_{gr}. The mass ejected in reaching coalescence is of order M_{12} for equal-mass binaries, and several times M_2 when $M_2 \ll M_1$. A black hole that grew to its current size through a succession of mergers should therefore have displaced a few times its own mass in stars. If this mass came mostly from stars that were originally in the nucleus, the density within r_h would drop drastically and the hardening would slow. Without some way of replenishing the supply of stars, decay could stall at a separation much greater than a_{gr}. This is the "final parsec problem": how to avoid stalling and bring the black holes from their separation of ~ 1 pc, when they first form a hard binary, to $\sim 10^{-2}$ pc, where gravity wave emission is efficient.

Changes in the binary's orbital eccentricity (equation 1.15) are potentially important because the gravity wave coalescence time drops rapidly as $e \to 1$ (equation 1.18). For a hard binary, scattering experiments give $K(e) \approx K_0 e(1-e^2)$, with $K_0 \approx 0.5$ for an equal-mass binary (Mikkola & Valtonen 1992; Quinlan 1996). The dependence of K on M_2/M_1 is not well understood. The implied changes in e as a binary decays from $a = a_h$ to a_{gr} are modest, $\Delta e \lesssim 0.2$, for all initial eccentricities. The rms change in the orientation of the binary's spin axis (equation 1.16) is $\delta\theta \approx 2(m_*/M_{12})^{1/2} \log^{1/2}(a_h/a)$, which is of order one degree or less if $m_* \approx M_\odot$. Reorientations of the binary's angular momentum affect the spin direction of the coalesced black hole and the direction of any associated jet (Merritt 2002).

1.4.2 Binary Black Holes in Galaxies

The scattering experiments summarized above treat the binary's environment as fixed and homogeneous. In reality, the binary is embedded at the center of an inhomogeneous and evolving galaxy, and the supply of stars that can interact with it is limited.

In a fixed spherical galaxy, stars can interact with the binary only if their pericenters lie within $\sim \mathcal{R} \times a$, where $\mathcal{R}$ is of order unity. Let $L_{lc} = \mathcal{R}a\sqrt{2[E-\Phi(\mathcal{R}a)]} \approx \sqrt{2GM_{12}\mathcal{R}a}$, the angular momentum of a star with pericenter $\mathcal{R}a$. The "loss cone" is the region in phase space defined by $L \leq L_{lc}$. The mass of stars in the loss cone is

$$\begin{aligned} M_{lc}(a) &= m_* \int dE \int_0^{L_{lc}} dL\, N(E,L^2) \\ &= m_* \int dE \int_0^{L_{lc}^2} dL^2 4\pi^2 f(E,L^2) P(E,L^2) \\ &\approx 8\pi^2 G M_{12} m_* \mathcal{R} a \int dE f(E) P_{rad}(E). \end{aligned} \tag{1.21}$$

Here P is the orbital period; in the final line, f is assumed isotropic and P has been approximated by the period of a radial orbit of energy E. An upper limit to the mass that is available to interact with the binary is $\sim M_{lc}(a_h)$, the mass within the loss cone when the binary first becomes hard; this is an upper limit since some stars that are initially within the loss cone will "fall out" as the binary shrinks. Assuming a singular isothermal sphere for the stellar distribution, $\rho \propto r^{-2}$, and taking the lower limit of the energy integral to be $\Phi(a_h)$, equation (1.21) implies

$$M_{lc}(a_h) \approx 3\mathcal{R}\mu. \tag{1.22}$$

We can compute the change in a that would result if the binary interacted with this entire mass, by using the fact the mean energy change of a star interacting with a hard binary is $\sim 3Gm_*\mu/2a$ (Quinlan 1996). Equating the energy carried away by stars with the change in the binary's binding energy gives

$$\frac{3}{2}\frac{G\mu}{a} dM \approx \frac{GM_1M_2}{2} d\left(\frac{1}{a}\right) \tag{1.23}$$

or

$$\ln\left(\frac{a_h}{a}\right) \approx 3\frac{\Delta M}{M_{12}} \approx \frac{9\mathcal{R}\mu}{M_{12}} \approx 9\mathcal{R}\frac{p}{(1+p)^2}, \qquad p \equiv M_2/M_1, \tag{1.24}$$

if ΔM is equated with M_{lc}. Only for very low mass ratios ($M_2 \lesssim 10^{-3}M_1$) is this decay factor large enough to satisfy equation (1.19), but the time required for such a small black hole to reach the nucleus is likely to exceed a Hubble time (Merritt 2000). Hence even under the most favorable assumptions, the binary would not be able to interact with enough mass to reach gravity-wave coalescence.

But the situation is even worse than this, since not all of the mass in the loss cone will find its way into the binary. The time scale for the binary to shrink is comparable with stellar orbital periods, and some of the stars with $r_{\rm peri} \approx a_h$ will only reach the binary after a has fallen below $\sim a_h$. We can account for the changing size of the loss cone by writing

$$\frac{dM}{dt} = \int_{E_0(t)}^{\infty} \frac{1}{P(E)} \frac{dM_{lc}}{dE} dE = 8\pi^2 G M_{12} m_* \mathcal{R} a(t) \int_{E_0(t)}^{\infty} f_i(E) dE, \tag{1.25}$$

where $M(t)$ is the mass in stars interacting with the binary and $f_i(E)$ is the initial distribution function; setting $P(E_0) = t$ reflects the fact that stars on orbits with periods less than t have already interacted with the binary and been ejected. Combining equations (1.23) and (1.25),

$$\frac{d}{dt}\left(\frac{1}{a}\right) \approx 24\pi^2 \mathcal{R} G m_* \int_{E_0(t)}^{\infty} f_i(E) dE. \tag{1.26}$$

Solutions to equation (1.26) show that a binary in a singular isothermal sphere galaxy stalls at $a_h/a \approx 2.5$ for $M_2 = M_1$, compared with $a_h/a \approx 10$ if the full loss cone were depleted (equation 1.24). In galaxies with shallower central cusps, decay of the binary would stall at even greater separations.

It is therefore entirely possible that uncoalesced binaries exist at the centers of many galaxies, particularly galaxies like large ellipticals with low central densities. But there is circumstantial evidence that long-lived black hole binaries are rare. Some radio galaxies show clear evidence of a recent flip in the black hole's spin orientation, as would occur when two black holes coalesce (Dennett-Thorpe et al. 2002), and their numbers are consistent with a coalescence rate that is roughly equal to the galaxy merger rate (Merritt & Ekers 2002). The almost complete lack of correlation between jet directions in Seyfert galaxies and the angular momenta of their disks (Ulvestad & Wilson 1984; Kinney et al. 2000) also suggests that black hole coalescences were common in the past. There are few if any "smoking gun" detections of binary black holes among AGN (e.g., Halpern & Eracleous 2000); the best, but still controversial, case is OJ 287 (Pursimo et al. 2000).

Below are discussed some additional mechanisms that have been proposed for extracting energy from binary black holes and hastening their decay. When the agent interacting with the binary is stars, continued decay of the binary implies continued destruction of the stellar cusp. However, for most of the mechanisms discussed below, the detailed effects on the stellar distribution have yet to be worked out.

1.4.2.1 Scattering of Stars into the Loss Cone

Destruction of a pre-existing stellar cusp generates strong gradients in the phase-space density at $L \approx L_{lc}$, the angular momentum of an orbit that lies just outside the loss cone. A small perturbation can deflect a star on such an orbit into the loss cone. This process has been studied in detail in the context of gravitational scattering of stars into the tidal disruption sphere of a single black hole (Lightman & Shapiro 1977; Cohn & Kulsrud

1978). Once a steady-state flow of stars into the loss cone has been set up, the distribution function near L_{lc} has the form:

$$f(E,L) \approx \frac{1}{\ln(1/R_{lc})}\overline{f}(E)\ln\left(\frac{R}{R_{lc}}\right) \tag{1.27}$$

(Lightman & Shapiro 1977), where $R \equiv L^2/L_c^2(E)$, $L_c(E)$ is the angular momentum of a circular orbit of energy E, and $\overline{f}$ is the distribution function far from the loss cone, assumed to be isotropic. The mass flow into the central object is $m_* \int \mathcal{F}(E)dE$, where

$$\mathcal{F}(E)dE = 4\pi^2 L_c^2(E)\left\{\oint \frac{dr}{v_r}\lim_{R\to 0}\frac{\langle(\Delta R)^2\rangle}{2R}\right\}\frac{\overline{f}}{\ln(1/R_{lc})}dE. \tag{1.28}$$

The quantity in brackets is the orbit-averaged diffusion coefficient in R. Yu (2002) evaluated the contribution of two-body scattering to the decay rate of a massive binary and found that it was usually too small to overcome the stalling that occurs when the loss cone is first emptied.

Standard loss cone theory assumes a quasi-steady-state distribution of stars in phase space near L_{lc}. This assumption is appropriate at the center of a globular cluster, where relaxation times are much shorter than the age of the universe, but is less appropriate for a galactic nucleus, where relaxation times almost always greatly exceed a Hubble time (e.g., Faber et al. 1997). The distribution function $f(E,L)$ immediately following the formation of a hard binary is approximately a step function,

$$f(E,L) \approx \begin{cases} \overline{f}(E), & L > L_{lc} \\ 0, & L < L_{lc}, \end{cases} \tag{1.29}$$

much steeper than the $\sim \ln L$ dependence in a collisionally relaxed nucleus (equation 1.27). Since the transport rate in phase space is proportional to the gradient of f with respect to L, steep gradients imply an enhanced flux into the loss cone. The total mass in stars consumed by the binary can exceed the predictions of the standard model by factors of a few, implying greater cusp destruction and more rapid decay of the binary (Milosavljević & Merritt 2003b). This time-dependent loss cone refilling might be particularly effective in a nucleus that continues to experience mergers or accretion events, in such a way that the loss cone repeatedly returns to an unrelaxed state with its associated steep gradients.

1.4.2.2 Re-ejection

Unlike the case of tidal disruption of stars by a single black hole, a star that interacts with a massive binary remains inside the galaxy and is available for further interactions. In principle, a single star can interact many times with the binary before being ejected from the galaxy or falling outside the loss cone; each interaction takes additional energy from the binary and hastens its decay. Consider a simple model in which a group of N stars in a spherical galaxy interacts with the binary and receives a mean energy increment of $\langle\Delta E\rangle$. Let the original energy of the stars be E_0. Averaged over a single orbital period $P(E)$, the binary hardens at a rate

$$\frac{d}{dt}\left(\frac{GM_1M_2}{2a}\right) = m_*\frac{N\langle\Delta E\rangle}{P(E)}. \tag{1.30}$$

In subsequent interactions, the number of stars that remain inside the loss cone scales as $L_{lc}^2 \propto a$, while the ejection energy scales as $\sim a^{-1}$. Hence $N\langle\Delta E\rangle \propto a^1 a^{-1} \propto a^0$. Assuming the singular isothermal sphere potential for the galaxy, one finds

$$\frac{a_h}{a(t)} \approx 1 + \frac{\mu}{M_{12}} \ln\left[1 + \frac{m_* N\langle\Delta E\rangle}{2\mu\sigma^2}\frac{t-t_h}{P(E_0)}\right] \tag{1.31}$$

(Milosavljević & Merritt 2003b). Hence the binary's binding energy increases as the logarithm of the time, even after all the stars in the loss cone have interacted at least once with the binary. Re-ejection would occur differently in nonspherical galaxies where angular momentum is not conserved and ejected stars could miss the binary on their second passage. However, there will generally exist a subset of orbits defined by a maximum pericenter distance $\lesssim a$, and stars scattered onto such orbits can continue to interact with the binary.

1.4.2.3 Chaotic Loss Cones

Loss cone dynamics are qualitatively different in non-axisymmetric (triaxial or bar-like) potentials, since a much greater number of stars may be on "centrophilic" orbits which take them near to the black hole(s) (Gerhard & Binney 1985). Triaxial models need to be taken seriously given recent demonstrations (Poon & Merritt 2002, 2004) that black hole nuclei can be stably triaxial and that the fraction of mass on centrophilic – typically chaotic – orbits can be large. The frequency of pericenter passages, $r_{\rm peri} < d$, for a chaotic orbit of energy E in a triaxial black hole nucleus is roughly linear in d, $N(r_{\rm peri} < d) \approx d \times A(E)$ (Merritt & Poon 2004). The total rate at which stars pass within a distance $\mathcal{R}a$ of the massive binary is therefore

$$\frac{dM}{dt} \approx \mathcal{R}a \int A(E) M_c(E) dE, \tag{1.32}$$

where $M_c(E)$ is the mass on centrophilic orbits. The implied feeding rate can be comparable to that in a spherical nucleus with a constantly refilled loss cone, and orders of magnitude greater than in a loss cone that is re-supplied via star-star interactions (Merritt & Poon 2004). Even transient departures from axisymmetry, for instance during mergers, might result in substantial loss cone refilling due to this mechanism.

1.4.2.4 Multiple Black Holes

If binary decay stalls, an uncoalesced binary may be present in a nucleus when a third black hole, or a second binary, is deposited there following a subsequent merger. The multiple black hole system that forms will engage in its own gravitational slingshot interactions, eventually ejecting one or more of the black holes from the nucleus (though probably not from the galaxy). This process has been extensively modeled assuming a fixed potential for the galaxy (e.g., Valtaoja et al. 1989; Mikkola & Valtonen 1990; Valtonen et al. 1994). The effect on the stellar distribution of $N \gtrsim 2$ interacting black holes is not well understood, although N-body simulations with $10-20$ massive particles and a "live" background show that the black holes displace ~ 10 times their own mass before being ejected (Merritt & Milosavljević 2003). The separation and eccentricity of the dominant binary can change dramatically during each interaction and this may be an effective way to shorten the gravity-wave coalescence time. In a wide, hierarchical triple, the eccentricity of the dominant binary oscillates through a maximum value of $\sim \sqrt{1-5\cos^2 i/3}$, $|\cos i| <$

$\sqrt{3/5}$, with i the mutual inclination angle (Kozai 1962). Blaes, Lee, & Socrates (2002) estimate that the coalescence times in equal-mass, hierarchical triples can be reduced by factors of ~ 10 via the Kozai mechanism.

1.4.2.5 Gas Dynamics

If the inner ~ 1 pc of the nucleus contains a mass in gas comparable to M_2, torques from the gas will cause the orbit of the smaller black hole to decay in a time of order the gas accretion time (Syer & Clarke 1995; Ivanov, Papaloizou, & Polnarev 1999). Given standard assumptions about accretion disk viscosities, the gas-dynamical decay rate would exceed that from gravity wave emission for $a > a_{acc}$, where

$$\frac{a_{acc}}{a_h} \approx 1 \times 10^{-3} \left(\frac{p}{0.1}\right)^{2/5} \left(\frac{\sigma}{200 \text{ km } s^{-1}}\right)^2 \tag{1.33}$$

(Armitage & Natarajan 2002). If the orbit of the secondary is strongly inclined with respect to the accretion disk around the larger black hole, its passages through the disk could generate periodic outbursts, and this has been suggested as a model for the ~ 12 yr cycle of optical flaring observed in the blazar OJ 287 (Lehto & Valtonen 1996). Gas deposition of the required magnitude almost certainly occurred during the quasar epoch, although it is less clear that this mechanism is effective for galaxies in the current universe.

1.5 *N*-Body Studies

Unless great care is taken, N-body studies of binary black hole dynamics are unlikely to give an accurate picture of the evolution expected in real galaxies. This follows from the result (§ 1.4.2) that time scales for two-body scattering of stars into the binary's loss cone are of order the Hubble time or somewhat longer in real galaxies. In N-body simulations, relaxation times are shorter by factors of $\sim N/10^{11}$ than in real galaxies, hence the long-term evolution of the binary is likely to be dominated by spurious loss cone refilling, Brownian motion of the black holes and other noise-driven effects (Milosavljević & Merritt 2003b). The stalling that is predicted in the absence of loss cone refilling in real galaxies (Fig. 1.4) can only be reproduced via N-body codes if the mean field is artificially smoothed and the black hole binary is "nailed down" (e.g., Quinlan & Hernquist 1997).

N-body studies are most useful at characterizing the early stages of binary formation and decay, or simulating the disruptive effects of a single black hole on an infalling galaxy; indeed scattering experiments (§ 1.4.1) are almost useless in these regimes. Due to algorithmic limitations, most N-body studies (e.g., Ebisuzaki, Makino, & Okumura 1991; Makino et al. 1993; Governato, Colpi, & Maraschi 1994; Makino & Ebisuzaki 1996; Makino 1997; Nakano & Makino 1999a,b; Hemsendorf, Sigurdsson, & Spurzem 2002) have been based on galaxy models with unrealistically large cores. The first N-body merger simulations using two black holes and realistically dense initial conditions (Merritt & Cruz 2001; see also Merritt et al. 2002) found that the black hole in the larger galaxy was efficient at tidally disrupting the steep cusp in the infalling galaxy, producing a remnant with only slightly higher central density than that of the giant galaxy initially. This result helps explain the absence of dense cusps in bright galaxies (Forbes, Franx, & Illingworth 1995) and the persistence of the "core fundamental plane" in the face of mergers (Holley-Bockelmann & Richstone 1999, 2000).

Quinlan & Hernquist (1997) studied the evolution of a black hole binary inside cuspy models with $\rho \sim r^{-1}$ and $\rho \sim r^{-2}$ and a range of black hole masses and particle numbers, $N \leq 2 \times 10^5$. Their N-body code was unable to simulate an actual merger and all of their detailed results were derived from initial conditions consisting of a single galaxy into which two "naked" black holes were symmetrically dropped. This artificial starting configuration resulted in substantial evolution of the cusp before the formation of the binary. The late evolution of the binary was found to be strongly dependent on N, due in part to spurious Brownian motion of the black hole particles. The cores that formed were characterized by strong velocity anisotropies, $\sigma_t \gg \sigma_r$, due to the ejection of stars on eccentric orbits.

Milosavljević & Merritt (2001) followed the evolution of cuspy ($\rho \sim r^{-2}$) galaxy models containing black holes, starting from pre-merger initial conditions and continuing until the binary separation had decayed a factor of ~ 10 below a_h. The initially steep nuclear cusps were converted to shallower, $\rho \sim r^{-1}$ profiles shortly after the black holes had formed a hard binary; thereafter the nuclear profile evolved slowly toward even shallower slopes. The decay rate of the binary was found not to be strongly dependent on N, probably due to the fact that the loss cone was continuously refilled by two-body scattering in these collisional simulations (Milosavljević & Merritt 2003b). The velocity anisotropies created during formation of the core were much milder than in the simulations of Quinlan & Hernquist (1997), and similar in magnitude to the anisotropies predicted by the adiabatic growth model (Fig. 1.1).

1.6 Observational Evidence for the Binary Black Hole Model

Figure 1.3 shows that a massive binary must eject of order its own mass in reaching a separation of a_{gr} if $M_2 \approx M_1$, or several times M_2 if $M_2 \ll M_1$. These numbers should be interpreted with caution since: (1) binaries might not decay as far as a_{gr}, or the final stages of decay might be driven by some process other than energy exchange with stars; (2) the definition of "ejection" used in Figure 1.3 is escape of a star from an isolated binary, and does not take into account the confining effect of the nuclear potential; (3) the effect of repeated mergers on nuclear density profiles, particularly mergers involving very unequal-mass binaries, is poorly understood. Nevertheless, a clear prediction of the binary black hole model is that galactic mergers should result in the removal of a mass of order M_{12} from the nucleus. As in the adiabatic growth model, we are handicapped in testing the theory by lack of knowledge of the primordial nuclear profiles. A reasonable guess is that all galaxies originally had steep, $\rho \sim r^{-2}$ density cusps, since these are generic in the low-luminosity ellipticals which are least likely to have been influenced by mergers, and since the adiabatic growth model predicts $\rho \sim r^{-2}$ (Fig. 1.1).

The "mass deficit" is defined as the difference in integrated mass between the observed density profile, and a $\rho \propto r^{-\gamma_0}$ profile extrapolated inward from the break radius r_b:

$$M_{def} \equiv 4\pi \int_0^{r_b} \left[\rho(r_b)\left(\frac{r}{r_b}\right)^{-\gamma_0} - \rho(r)\right] r^2 dr. \tag{1.34}$$

Milosavljević et al. (2002) and Ravindranath, Ho, & Filippenko (2002) computed M_{def} in samples of "core"-profile elliptical galaxies. The former authors found a good proportionality between M_{def} and $M_\bullet$, with $\langle M_{def}/M_\bullet \rangle \approx 1$ for $\gamma_0 = 1.5$ and $\langle M_{def}/M_\bullet \rangle \approx 10$ for $\gamma_0 = 2$ (Fig. 1.4). These numbers are within the range predicted by the binary black hole model, given the uncertainties associated with multiple mergers. Ravindranath et al. (2002) computed black hole masses using a shallower assumed slope for the $M_\bullet - \sigma$ relation, $M_\bullet \propto \sigma^{3.75}$, and found a steeper, nonlinear dependence of M_{def} on $M_\bullet$.

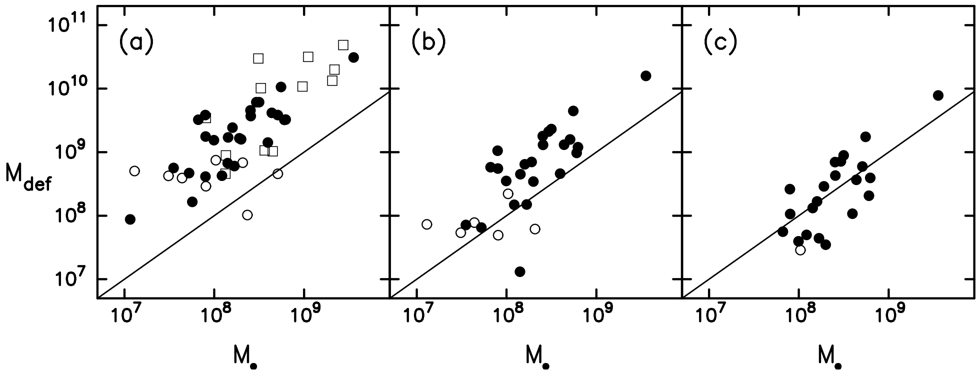

Fig. 1.4. Mass deficit versus black hole mass for three different assumed values of γ_0, the logarithmic slope of the density cusp before energy input from the black holes: (*a*) $\gamma_0 = 2$, (*b*) $\gamma_0 = 1.75$, and (*c*) $\gamma_0 = 1.5$. Units are solar masses. Solid lines are $M_{def} = M_\bullet$ (Adapted from Milosavljević et al. 2002).

More rigorous tests of the binary black hole model will require a better understanding of the expected form of $\rho(r)$. As discussed above, while the best current N-body simulations suggest $\rho \sim r^{-1}$ following binary formation (Milosavljević & Merritt 2001), the simulations are dominated by noise over the long term. If the decay stalls, the predicted density profile can be very different: a "hole" forms inside of $\sim 3a_{stall}$ (e.g., Fig. 1 of Zier & Biermann 2001). Central minima may in fact have been seen in the luminosity profiles of a few galaxies (Lauer et al. 2002).

Other processes could result in energy exchange between binary black holes and stars. If the binary eventually coalesces, the gravitational radiation carries a *linear* momentum leading to a recoil of the coalesced black hole at a velocity $\upsilon_{\rm recoil} \sim 10^2 - 10^3$ km s^{-1} (Bekenstein 1973; Fitchett 1983; Eardley 1983), and possibly even higher if the black holes were rapidly spinning prior to coalescence (Redmount & Rees 1989). Recoil velocities of this order would eject the black hole from the center of the nucleus and its subsequent infall would displace stars. Quantitative evaluation of this effect will require more accurate estimates of $\upsilon_{\rm recoil}$ based on fully general-relativistic calculations of black hole coalescence.

A major focus of future work should be to calculate the evolution of $\rho(r)$ as predicted by the various scenarios for binary decay discussed in this article.

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References

Armitage, P. J., & Natarajan, P. 2002, ApJ, 567, L9

Baranov, A. S. 1984, Astron. Zh., 61, 1098

Begelman, M. C., Blandford, R. D., & Rees, M. J. 1980, Nature, 287, 307

Bekenstein, J. 1973, ApJ, 183, 675

Blaes, O., Lee, M. H., & Socrates, A. 2002, ApJ, 578, 775

Cavaliere, A., & Vittorini, V. 2000, ApJ, 543, 599

Cipollina, M. 1995, A&AS, 110, 155

Cipollina, M., & Bertin, G. 1994, A&A, 288, 43
Cohn, H., & Kulsrud, R. 1978, ApJ, 226, 1087
Dennett-Thorpe, J., Scheuer, P. A. G., Laing, R. A., Bridle, A. H., Pooley, G. G., & Reich, W. 2002, MNRAS, 330, 609
Eardley, D. M. 1985, in Gravitational Radiation, ed. N. Deruelle & T. Piran (Amsterdam: North-Holland), 257
Ebisuzaki, T., Makino, J., & Okumura, S. K. 1991, Nature, 354, 212
Faber, S. M., et al. 1997, AJ, 114, 1771
Fan, X., et al. 2001, AJ, 122, 2833
Forbes, D. A., Franx, M., & Illingworth, G. D. 1995, AJ, 109, 1988
Fitchett, M. 1983, MNRAS, 203, 1049
Gerhard, O. E., & Binney, J. 1985, MNRAS, 216, 467
Gondolo, P., & Silk, J. 1999, Phys. Rev. Lett., 83, 1719
Goodman, J., & Binney, J. 1984, MNRAS, 207, 511
Governato, F., Colpi, M., & Maraschi, L. 1994, MNRAS, 271, 317
Halpern, J. P., & Eracleous, M. 2000, ApJ, 531, 647
Hemsendorf, M., Sigurdsson, S., & Spurzem, R. 2002, ApJ, 581, 1256
Hills, J. G. 1983, AJ, 88, 1269
——. 1992, AJ, 103, 1955
Hills, J. G., & Fullerton, L. W. 1980, AJ, 85, 1281
Holley-Bockelmann, K., & Richstone, D. 1999, ApJ, 517, 92
——. 2000, ApJ, 531, 232
Ivanov, P. B., Papaloizou, J. C. B., & Polnarev, A. G. 1999, MNRAS, 307, 79
Kinney, A. L., Schmitt, H. R., Clarke, C. J., Pringle, J. E., Ulvestad, J. S., & Antonucci, R. R. J. 2000, ApJ, 537, 152
Kozai, Y. 1962, AJ, 67, 591
Lauer, T. R., et al. 2002, AJ, 124, 1975
Lee, M. H., & Goodman, J. 1989, ApJ, 343, 594
Leeuwin, F., & Athanassoula, E. 2000, MNRAS, 317, 79
Lehto, H. J., & Valtonen, M. J. 1996, ApJ, 460, 207
Lightman, A. P., & Shapiro, S. L. 1977, ApJ, 211, 244
Lynden-Bell, D. 1963, The Observatory, 83, 23
Makino, J. 1997, ApJ, 478, 58
Makino, J., & Ebisuzaki, T. 1996, ApJ, 465, 527
Makino, J., Fukushige, T., Okumura, S. K., & Ebisuzaki, T. 1993, PASJ, 45, 303
Merritt, D. 2000, in Dynamics of Galaxies, ed. F. Combes, G. A. Mamon, & V. Charmandaris, (San Francisco: ASP), 221
——. 2001, ApJ, 556, 245
——. 2002, ApJ, 568, 998
Merritt, D., & Cruz, F. 2001, ApJ, 551, L41
Merritt, D., & Ekers, R. D. 2002, Science, 297, 1310
Merritt, D., & Milosavljević, M. 2003, DARK 2002: 4th International Heidelberg Conference on Dark Matter in Astro and Particle Physics, ed. H. V. Klapdor-Kleingrothaus & R. Viollier (astro-ph/0205140)
Merritt, D, Milosavljević, M., Verde, L., & Jimenez, R. 2002, Phys. Rev. Lett., 88, 191301
Merritt, D., & Poon, M. Y. 2004, in preparation
Merritt, D., & Quinlan, G. D. 1998, ApJ, 498, 625
Mikkola, S., & Valtonen, M. J. 1990, ApJ, 348, 412
——. 1992, MNRAS, 259, 115
Milosavljević, M., & Merritt, D. 2001, ApJ, 563, 34
——. 2003a, to appear in Proceedings of the 4th LISA Symposium (astro-ph/0212270)
——. 2003b, ApJ, in press (astro-ph/0212459)
Milosavljević, M., Merritt, D., Rest, A., & van den Bosch, F. C. 2002, MNRAS, 331, L51
Nakano, T., & Makino, J. 1999a, ApJ, 510, 155
——. 1999b, ApJ, 525, L77
Peebles, P. J. E. 1972, Gen. Rel. Grav., 3, 61
Peters, P. C. 1964, Phys. Rev. B, 136, 1224
Poon, M., & Merritt, D. 2002, ApJ, 568, L89
——. 2004, ApJ, in press (astro-ph/0212581)

Power, C., Navarro, J. F., Jenkins, A., Frenk, C. S., White, S. D. M., Springel, V., Stadel, J., & Quinn, T. 2003, MNRAS, 338, 14
Pursimo, T., et al. 2000, A&AS, 146, 141
Quinlan, G. D. 1996, New Astron., 1, 35
Quinlan, G. D., & Hernquist, L. 1997, New Astron., 2, 533
Quinlan, G. D., Hernquist, L., & Sigurdsson, S. 1995, ApJ, 440, 554
Ravindranath, S., Ho, L. C., & Filippenko, A. V. 2002, ApJ, 566, 801
Redmount, I. H., & Rees, M. J. 1989, Comm. Astrophys., 14, 165.
Roos, N. 1981, A&A, 104, 218
Saslaw, W. C., Valtonen, M. J., & Aarseth, S. J. 1974, ApJ, 190, 253
Syer, D., & Clarke, C. J. 1995, MNRAS, 277, 758
Ulvestad, J. S., & Wilson, A. S. 1984, ApJ, 285, 439
Valtaoja, L., Valtonen, M. J., & Byrd, G. G. 1989, ApJ, 343, 47
Valtonen, M. J., Mikkola, S., Heinamaki, P., & Valtonen, H. 1994, ApJS, 95, 69
van der Marel, R. P. 1999, AJ, 117, 744
Young, P. J. 1980, ApJ, 242, 1232
Yu, Q. 2002, MNRAS, 331, 935
Zier, C., & Biermann, P. L. 2001, A&A, 377, 23